Anyon Condensation and Continuous Topological Phase Transitions in Non-Abelian Fractional Quantum Hall States

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We find a series of possible continuous quantum phase transitions between fractional quantum Hall (FQH) states at the same filling fraction in two-component quantum Hall systems. These can be driven by tuning the interlayer tunneling and/or interlayer repulsion. One side of the transition is the Halperin $(p, p, p-3)$ Abelian two-component state while the other side is the non-Abelian Z_4 parafermion (Read-Rezayi) state. We predict that the transition is a continuous transition in the 3D Ising class. The critical point is described by a Z_2 gauged Ginzburg-Landau theory. These results have implications for experiments on two-component systems at $\nu = 2/3$ and single-component systems at $\nu = 8/3$.

One of the most challenging problems in the study of quantum many-body systems is to understand transitions between topologically ordered states [1]. Since topological states cannot be characterized by broken symmetry and local order parameters, we cannot use the conventional Ginzburg-Landau theory. When non-Abelian topological states are involved, the transitions that are currently understood are essentially all equivalent to the transition between weak and strong-paired BCS states [2, 3]. Over the last ten years, while there has been much work on the subject, there has not been another quantum phase transition in a physically realizable system, involving a non-Abelian phase, for which we can answer the most basic questions of whether the transition can be continuous and what the critical theory is. Here we present an additional example in the context of fractional quantum Hall (FQH) systems.

The quasiparticle excitations in FQH states carry fractional statistics and fractional charge [1]. In particular, in a (ppq) bilayer FQH state [1, 4], there is a type of excitation, called a fractional exciton (f-exciton), which is a bound state of a quasiparticle in one layer and an oppositely charged quasihole in the other layer. It carries fractional statistics. As we increase the repulsion between the electrons in the two layers, the energy gap of the f-exciton will be reduced; when it is reduced to zero, the f-exciton will condense and drive a phase transition. When the anyon number has only a mod n conservation, this can even lead to a non-Abelian FQH state [2], yet little is known about "anyon condensation" [5–8]. A better understanding of these phase transitions may aid the quest for experimental detection of non-Abelian FQH states, because one side of the transition – in our case the (330) state at $\nu = 2/3$ – can be accessed experimentally [9, 10]. The results of this paper suggest a new way of experimentally tuning to a non-Abelian state in bilayer FQH states, similar to the transition from the (331) state to the Moore-Read Pfaffian at $\nu = 1/2$ [2, 3, 11].

In the (ppq) state, when the energy gap of the f-exciton at $\boldsymbol{k} = 0$ is reduced to zero, the f-exciton will condense[2]. The transition can be described by the $\phi = 0 \to \phi \neq 0$ transition in a Ginzburg-Landau theory with a Chern-Simons (CS) term: $\mathcal{L} = |(\partial_0 + ia_0)\phi|^2 - v^2|(\partial_i + ia_i)\phi|^2 - f|\phi|^2 - g|\phi|^4 - \frac{\pi}{\theta}\frac{1}{4\pi}a_\mu\partial_\nu a_\lambda \epsilon^{\mu\nu\lambda}$, where θ is the statistical angle of the f-exciton. Such a transition changes the Abelian (ppq) FQH state to another Abelian charge-$2e$ FQH state.[2, 3]

In the presence of interlayer electron tunneling, the number of f-excitons is conserved only mod $p - q$. A new term $\delta\mathcal{L} = t(\phi\hat{M})^{p-q} + h.c.$ must be included, where $\hat{M}$ is an operator that creates 2π flux of the $U(1)$ gauge field a_μ. With this new term, what is the fate of the $\phi = 0 \to \phi \neq 0$ transition?

When $p - q = 2$, the f-excitons happen to be fermions (*ie* $\theta = \pi$), so we can map the $\mathcal{L} + \delta\mathcal{L}$ theory to a free fermion theory and solve the problem.[2] The problem is closely related to the transition from weak to strong-pairing of a $p_x + ip_y$ paired BCS superconductor [3]. The interlayer electron tunneling splits the single continuous transition between the $(p, p, p-2)$ and the charge-$2e$ FQH states into two continuous transitions. The new phase between the two new transitions is the non-Abelian Pfaffian state [12]. This is the only class of phase transition involving a non-Abelian FQH state for which anything is known.

When $p - q \neq 2$, the f-excitons are anyons. The problem becomes so hard that we do not even know where to start. But we may guess that even when $p - q \neq 2$, an interlayer electron tunneling may still split the transition between the (ppq) and charge-$2e$ FQH states. The new phase between the two new transitions may be a non-Abelian FQH state [2]. When $p - q = 3$, it was suggested that the new phase is a Z_4 parafermion (Read-Rezayi[13]) FQH state [14]. This is because anti-symmetrizing the (330) wave function between the coordinates of the two layers yields the Z_4 parafermion wave function, in direct analogy to the known continuous transition from (331) to Pfaffian, where anti-symmetrizing the (331) wave function yields the single-layer Pfaffian wave function.

In this letter, we show that *the Abelian $(p, p, p-3)$ state can change into the Z_4 parafermion state through a continuous quantum phase transition. The transition is*

in the 3D Ising class. The critical point is described by a Z_2 gauged Ginzburg-Landau theory.

These results are experimentally relevant in the case $p = 3$. The (330) state has been experimentally realized in double layer and wide quantum wells [10]. The existence of a neighboring single layer non-Abelian state in the phase diagram, which can be realized by tuning the interlayer tunneling/repulsion, suggests that experiments have a chance of realizing this transition. Furthermore, recent detailed experimental studies of the energy gaps of the $\nu = 8/3$ FQH state in single-layer systems indicate that it might be an exotic state, as opposed to a conventional Laughlin or hierarchy state [15]. This observation, together with the result of this paper that the Z_4 parafermion state lies close to the experimentally observed (330) state in the quantum Hall phase diagram, suggests that the Z_4 parafermion state ought to be considered as a candidate – in addition to other proposed possibilities (e.g. [16]) – in explaining the plateau at $\nu = 8/3$. In the case of the 5/2 plateau, numerical studies have recently suggested that finite layer thickness of the quantum well may stabilize the non-Abelian Pfaffian state [17], while it is also known that the Pfaffian state is near to the (331) state in the phase diagram; similarly, the results here suggest that finite layer thickness may also help stabilize the Z_4 parafermion state at $\nu = 8/3$ because of its proximity in the phase diagram to the (330) state.

The conceptual breakthrough in our understanding is a recently discovered low energy effective theory for the Z_4 parafermion state, which was found to be a $U(1)\times U(1) \rtimes Z_2$ Chern-Simons (CS) theory (a $U(1) \times U(1)$ CS theory coupled with a Z_2 gauge symmetry).[18] This is closely related to the effective theory for the $(p, p, p-3)$ state, which is a $U(1)\times U(1)$ CS theory. So effective theories for the Z_4 and the $(p, p, p-3)$ states only differ by a Z_2 gauge symmetry. Thus the transition between the $(p, p, p-3)$ and the Z_4 states may just be a Z_2 "gauge symmetry-breaking" transition induced by the condensation of a Z_2 charged field. We find that the $U(1) \times U(1) \rtimes Z_2$ CS theory contains a certain electrically neutral, bosonic quasiparticle that carries a Z_2 gauge charge. We argue that this bosonic quasiparticle becomes gapless at the transition and its condensation breaks the Z_2 gauge symmetry and yields the $(p, p, p-3)$ state.

To obtain the above results, without losing generality, let us choose $p = 3$, and consider the (330) state and the corresponding filling fraction $\nu = 2/3$ Z_4 parafermion state. The same results would also apply to filling fractions $\nu = 2n + 2/3$, where n is an integer. We begin by explaining the quasiparticle content of the Z_4 states, then we show that there exists an electrically neutral bosonic quasiparticle in the Z_4 state whose condensation yields the (330) state and that carries a Z_2 gauge charge in the low energy effective theory. Finally we discuss some consequences for physically measurable quantities.

One way to understand the topologically inequivalent excitations is through ideal wave functions, which admit a great variety of powerful tools for analysis of their physical properties [12, 13, 19–25]. In the ideal wave function approach, the ground state and quasiparticle wave functions of a FQH state are taken to be correlation functions of a 2D CFT: $\Phi_\gamma(\{z_i\}) \sim \langle V_\gamma(0) \prod_{i=1}^N V_e(z_i)\rangle$, where Φ_γ is a wave function with a single quasiparticle of type γ located at the origin and $z_i = x_i + iy_i$ is the coordinate of the ith electron. V_γ is a quasiparticle operator in the CFT and V_e are electron operators. The electron operator, through its operator product expansions (OPE), forms the chiral algebra of the CFT. Quasiparticles correspond to representations of the chiral algebra. Two operators V_γ and $V_{\gamma'}$ correspond to topologically equivalent quasiparticles if they differ by electron operators.

The Z_4 parafermion states, which exist at $\nu = 2/(2M+1)$ have $5(2M+1)$ topologically distinct quasiparticles. These can be organized into three representations of a magnetic translation algebra,[21] which each contain $2(2M+1)$, $2(2M+1)$, and $2M+1$ quasiparticles – see Table I, where we also listed a representative operator in the corresponding CFT description of these states. The CFT description of these states is formulated in terms of the Z_4 parafermion CFT [26] and a free boson CFT. The Z_4 parafermion CFT can be formulated in terms of an $SU(2)_4/U(1)$ coset CFT [27] or, equivalently, as the theory of a scalar boson φ_r, compactified at a special radius $R^2 = 6$ so that $\varphi_r \sim \varphi_r + 2\pi R$, and that is gauged by a Z_2 action: $\varphi_r \sim -\varphi_r$ [28]. Such a CFT is called the $U(1)/Z_2$ orbifold CFT. In Table I, we have included labellings of the operators in the CFT using both of these formulations.

The fusion rules of these quasiparticles can be obtained from the fusion rules of the Z_4 parafermion CFT: $\Phi^0_a \times \Phi^l_m = \Phi^l_{m+a}$, $\Phi^1_1 \times \Phi^1_1 = \Phi^0_2 + \Phi^2_2$, and $\Phi^2_2 \times \Phi^2_2 = \mathbb{I} + \Phi^0_4 + \Phi^2_0$. Φ^l_m exists for $l + m$ even, $0 \le l \le n$, and is subject to the following equivalences: $\Phi^l_m \sim \Phi^l_{m+2n} \sim \Phi^{n-l}_{m-n}$, where $n = 4$ for the Z_4 parafermion CFT.

Another useful way to understand the topological order of the Z_4 FQH state is through its bulk effective field theory, for which there are several different formulations.[18, 29, 30] Here we use the $U(1)\times U(1)\rtimes Z_2$ CS theory.[18] This is the theory of two $U(1)$ gauge fields, a and $\tilde{a}$, with an additional Z_2 gauge symmetry that corresponds to interchanging a and $\tilde{a}$ at any given point in space. The Lagrangian is $\mathcal{L} = \frac{p}{4\pi}(a\partial a + \tilde{a}\partial\tilde{a}) + \frac{q}{4\pi}(a\partial\tilde{a} + \tilde{a}\partial a)$, where $a\partial a$ is shorthand for $\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$. For $p - q = 3$, it was found that the ground state degeneracy on genus g surfaces of this theory agrees with that of the Z_4 parafermion FQH states at $\nu = 2/(2p - 3)$. In addition, it was found that the Z_2 vortices, which correspond to defects around which a and $\tilde{a}$ are interchanged, correspond to the non-Abelian Φ^1_m quasiparticles (**6 − 11** in Table I).[18] It was also found that quasiparticle **3**, which corresponds to the operator Φ^0_4 (see Table I), is

	$\Phi^l_m e^{iQ\sqrt{\nu^{-1}}\varphi_c}$	Z_2 Orbifold Label	$\{n_l\}$	$h_{pf}+h_{ga}$
0	$\mathbb{I}\sim\Phi^0_2 e^{i\sqrt{3/2}\varphi_c}$	$\mathbb{I}\sim\phi^1_N e^{i\sqrt{3/2}\varphi_c}$	1 1 1 0 0 1	0+0
1	$e^{i2/3\sqrt{3/2}\varphi_c}$	$e^{i2/3\sqrt{3/2}\varphi_c}$	1 1 1 1 0 0	$0+\frac{1}{3}$
2	$e^{i4/3\sqrt{3/2}\varphi_c}$	$e^{i4/3\sqrt{3/2}\varphi_c}$	0 1 1 1 1 0	$0+\frac{4}{3}$
3	Φ^0_4	$j_r\sim\partial\varphi_r$	0 0 1 1 1 1	1+0
4	$\Phi^0_4 e^{i2/3\sqrt{3/2}\varphi_c}$	$j_r e^{i2/3\sqrt{3/2}\varphi_c}$	1 0 0 1 1 1	$1+\frac{1}{3}$
5	$\Phi^0_4 e^{i4/3\sqrt{3/2}\varphi_c}$	$j_r e^{i4/3\sqrt{3/2}\varphi_c}$	1 1 0 0 1 1	$1+\frac{4}{3}$
6	$\Phi^1_1 e^{i1/6\sqrt{3/2}\varphi_c}$	$\sigma_1 e^{i1/6\sqrt{3/2}\varphi_c}$	1 1 0 1 0 1	$\frac{1}{16}+\frac{1}{48}$
7	$\Phi^1_1 e^{i5/6\sqrt{3/2}\varphi_c}$	$\sigma_1 e^{i5/6\sqrt{3/2}\varphi_c}$	1 1 1 0 1 0	$\frac{1}{16}+\frac{25}{48}$
8	$\Phi^1_1 e^{i9/6\sqrt{3/2}\varphi_c}$	$\sigma_1 e^{i9/6\sqrt{3/2}\varphi_c}$	0 1 1 1 0 1	$\frac{1}{16}+\frac{27}{16}$
9	$\Phi^1_5 e^{i1/6\sqrt{3/2}\varphi_c}$	$\tau_1 e^{i1/6\sqrt{3/2}\varphi_c}$	1 0 1 1 1 0	$\frac{9}{16}+\frac{1}{48}$
10	$\Phi^1_5 e^{i5/6\sqrt{3/2}\varphi_c}$	$\tau_1 e^{i5/6\sqrt{3/2}\varphi_c}$	0 1 0 1 1 1	$\frac{9}{16}+\frac{25}{48}$
11	$\Phi^1_5 e^{i9/6\sqrt{3/2}\varphi_c}$	$\tau_1 e^{i9/6\sqrt{3/2}\varphi_c}$	1 0 1 0 1 1	$\frac{9}{16}+\frac{27}{16}$
12	$\Phi^2_2 e^{i1/3\sqrt{3/2}\varphi_c}$	$\cos(\frac{\varphi_r}{\sqrt{6}})e^{i1/3\sqrt{3/2}\varphi_c}$	1 0 1 1 0 1	$\frac{1}{12}+\frac{1}{12}$
13	$\Phi^2_2 e^{i\sqrt{3/2}\varphi_c}$	$\cos(\frac{\varphi_r}{\sqrt{6}})e^{i\sqrt{3/2}\varphi_c}$	1 1 0 1 1 0	$\frac{1}{12}+\frac{3}{4}$
14	$\Phi^2_2 e^{i5/3\sqrt{3/2}\varphi_c}$	$\cos(\frac{\varphi_r}{\sqrt{6}})e^{i5/3\sqrt{3/2}\varphi_c}$	0 1 1 0 1 1	$\frac{1}{12}+\frac{25}{12}$

TABLE I: Quasiparticles in the Z_4 parafermion FQH state at $\nu = 2/3$ ($M = 1$). The different representations of the magnetic translation algebra[21] are separated by horizontal lines. Q is the electric charge and h_{pf} and h_{ga} are the scaling dimensions of the Z_4 parafermion field Φ^l_m and the $U(1)$ vertex operator $e^{i\alpha\varphi_c}$, respectively. φ_c is a free scalar boson that describes the charge sector. $\{n_l\}$ is the occupation number sequence associated with the quasiparticle pattern of zeros.

charged under the Z_2 gauge symmetry. This latter result is suggested by the orbifold formulation of the Z_4 parafermion CFT [28], where Φ^0_4 corresponds to a $U(1)$ current $j_r \sim \partial\varphi_r$. In the Z_2 orbifold, the scalar boson φ_r is gauged by the Z_2 action $\varphi_r \sim -\varphi_r$. The Z_2 gauge symmetry in the bulk CS theory is the Z_2 gauging in the orbifold CFT, which suggests that the quasiparticle Φ^0_4 would carry a Z_2 gauge charge.

From Table I, we see that the Z_4 states contain a special quasiparticle, **3** (Φ^0_4), which is electrically neutral, fuses with itself to the identity, and has Abelian fusion rules with all other quasiparticles. Φ^0_4 has scaling dimension 1 and is a bosonic operator. In the following we show that the condensation of this neutral bosonic quasiparticle yields the topological order of the (330) phase. Before condensation, two excitations are topologically equivalent if they differ by an electron, which is a local excitation. After condensation, all allowed quasiparticles must be local with respect to both the electron and Φ^0_4, and two quasiparticles will be topologically equivalent if they differ either by an electron or by Φ^0_4. In the CFT language, this means that Φ^0_4 has been added to the chiral algebra and will appear in the Hamiltonian. Such a situation was analyzed in a general mathematical setting for topological phases in Ref. 5.

From the OPE of Φ^0_4 and the other quasiparticle operators in the CFT description, we find that Φ^0_4 is mutually local with respect to the quasiparticles in the first and third representations of the magnetic algebra, which consist of the quasiparticles made of Φ^0_m and Φ^2_m (see Table I). However, its mutual locality exponent with the Φ^1_m quasiparticles is half-integer, which means that Φ^0_4 is non-local with respect to those quasiparticles. This is expected, because the Φ^1_m quasiparticles were found to correspond to Z_2 vortices in the $U(1)\times U(1)\rtimes Z_2$ CS theory while Φ^0_4 was found to carry Z_2 charge. Thus we would expect that Φ^0_4 would be non-local with respect to the Φ^1_m quasiparticles, with a half-integer mutual locality exponent. As a result, quasiparticles **6** − **11** are no longer valid (particle-like) topological excitations after condensation.

Since quasiparticles that differ by Φ^0_4 are regarded as topologically equivalent after condensation, quasiparticles **0**, **1**, and **2** become topologically equivalent to quasiparticles **3**, **4**, and **5** (see Table I), leaving three topologically distinct quasiparticles from this representation. Furthermore, the three quasiparticles in the third representation split into 6 topologically distinct quasiparticles. The reason for this was discussed in Ref. 5. Consider the fusion of a Φ^2_2 quasiparticle, which we will label as $\boldsymbol{\gamma}$, and its conjugate: $\boldsymbol{\gamma}\times\bar{\boldsymbol{\gamma}} = \mathbf{0}+\mathbf{3}+\mathbf{13}$. After condensation, we identify **3** (Φ^0_4) with the vacuum sector, so if $\boldsymbol{\gamma}$ does not split into at least two different quasiparticles, then there would be two different ways for it to annihilate into the vacuum with its conjugate. A basic property of topological phases is that particles annihilate into the vacuum in a unique way, so $\boldsymbol{\gamma}$ must split into at least two different quasiparticles. Since the quantum dimension of $\boldsymbol{\gamma}$ is 2, it must split into exactly two quasiparticles, each with quantum dimension 1: $\boldsymbol{\gamma}\to\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2$. $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are now Abelian quasiparticles because they have unit quantum dimension. Therefore, we see that the 15 quasiparticles in the Z_4 parafermion state become, after condensation of Φ^0_4, the 9 Abelian quasiparticles of the (330) state.

As the energy gap to quasiparticle **3** (Φ^0_4) is reduced, the low energy effective theory will simply be the theory of this bosonic field coupled to a Z_2 gauge field. The phase transition is a Higgs transition of this Z_2 charged boson (at least in the case where we explicitly break the global Z_2 symmetry of layer exchange). Note that there is no $U(1)$ symmetry that conserves the density of this kind of excitation because Φ^0_4 can annihilate with itself into the vacuum. Such a theory of a real scalar coupled to a Z_2 gauge field was studied in Ref. 31, and it was found that the transition is continuous and in the 3D Ising universality class. As we mentioned before, the same transition, when viewed as a transition from the (330) state to the Z_4 state, is induced by an anyon condensation with a mod-3 conservation. This suggests that the anyon condensation can, surprisingly, be described by a Z_2-charged boson condensation.

Given this result, a natural question is why fluctuations of Φ^0_4 should physically be related to interlayer density

fluctuations, which can be tuned by the interlayer tunneling and interlayer repulsion. One answer to this question comes from the analysis of the wave functions. The (330) wave function in real space is $\Phi_{(330)}(\{z_i, w_i\}) = \langle 0|\prod_{i=1}^{N}\psi_{e1}(z_i)\psi_{e2}(w_i)|330\rangle$, where ψ_{ei} is the electron annihilation operator in the ith layer. When interlayer tunneling is increased, the gap between the single-particle symmetric and anti-symmetric states is increased, until eventually all electrons occupy the symmetric set of orbitals, created by ψ_{e+}, where $\psi_{e\pm} \propto \psi^\dagger_{e1} \pm \psi^\dagger_{e2}$. The natural wave function to guess in the limit of infinite interlayer tunneling is thus the projection onto the symmetric states: $\Phi(\{z_i\}) = \langle 0|\prod_{i=1}^{2N}(\psi_{e1}(z_i)+\psi_{e2}(z_i))|330\rangle$, which in this case is the Z_4 parafermion wave function [14]. Thus starting from the (330) state, the wave function analysis suggests that interlayer tunneling will yield the Z_4 parafermion state. Since the (330) state can be obtained from the Z_4 parafermion state by condensing Φ^0_4, we are led to take this wave function analysis seriously and are led to conclude that the gap to Φ^0_4 can be controlled by interlayer tunneling. Note that tuning interlayer tunneling will tune fluctuations of the operator $\psi^\dagger_{e+}\psi_{e-}$, which is related to interlayer density fluctuations (since $\psi^\dagger_{e+}\psi_{e-} + h.c. = \psi^\dagger_{e1}\psi_{e1} - \psi^\dagger_{e2}\psi_{e2}$). Since the dimensionless parameters that we are concerned with are t/V_{inter} and V_{inter}/V_{intra}, we see that properly tuning the inter/intra -layer Coulomb repulsions (given by $V_{inter/intra}$) should also drive the interlayer density fluctuations and the corresponding transition. For yet a different perspective, we refer to[32].

Since the transition between these two FQH states is driven by the condensation of a neutral quasiparticle, it will be difficult to observe in experiments. Experiments have observed a phase transition in bilayer systems at $\nu = 2/3$ by measuring the kink in the charged excitation gap in charge transport measurements, though that may be an indication of a different transition [10]. One can also observe the (330) to Z_4 parafermion transition directly by probing the gapless neutral mode. The analysis here predicts that the bulk of the sample should remain an electrical insulator but become a thermal conductor at the transition. Furthermore, as the transition is approached, the fluctuations in the operator $\psi^\dagger_{e-}\psi_{e+}$ should correspond to fluctuating electric dipole moments between the two layers, which can be probed through surface acoustic waves.[33] Alternatively, experiments measuring minimal quasiparticle charge would be able to detect this transition, because the Z_4 state at $\nu = 2/3$ has a minimal quasiparticle charge of $e/6$, while the (330) state has a minimal quasiparticle charge of $e/3$.

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